# Moser lower bounds

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# Chapter 1

# Introduction

note: this blueprint was drafted by Claude

*Moser's worm problem* asks for the convex region of smallest area in the plane that contains a congruent copy of every rectifiable curve of unit length (a *worm*). Any such region is called a *Moser set*. The exact optimum is unknown; the best published upper and lower bounds sandwich it in a narrow interval. This project pursues a *computer-checked* lower bound for the area of a Moser set by discretising the space of candidate convex hulls and the space of planar isometries, and then verifying that no sufficiently small polygon can contain every unit worm.

The strategy is iterative pruning of a *working set* of convex candidate polygons: polygons are dropped when they are too large, when another candidate is a subset, or when a new worm fails to embed inside them. At termination the working set is empty, which (modulo the discretisation error bookkeeping) rules out all Moser sets below the area threshold. We currently target the threshold 0.232240.

# Chapter 2

# Rational planar geometry

All geometry in this project is carried out over $\mathbb{Q}^2$ so that decidable computations can be used throughout. Real-valued errors enter only through explicit approximation bounds.

**Definition 1** (Rational point)**.** A *rational point* is an element of $\mathbb{Q}^2$, represented as a function $\mathrm{Fin}\,2 \to \mathbb{Q}$.

**Definition 2** (Cross product, dot product, squared length)**.** For $u, v \in \mathbb{Q}^2$ we define $u \times v = u_0 v_1 - u_1 v_0$, $u \cdot v = u_0 v_0 + u_1 v_1$, and $\|v\|^2 = v_0^2 + v_1^2$.

**Lemma 3** (Squared length is nonnegative, and positive on nonzero vectors)**.** *For all* $v \in \mathbb{Q}^2$, $\|v\|^2 \geq 0$, *with strict inequality when* $v \neq 0$.

**Definition 4** (Counterclockwise rotation by a right angle)**.** The map $R : (x, y) \mapsto (-y, x)$ is a $90^\circ$ counterclockwise rotation, and preserves squared length.

**Definition 5** (Closed / open half-spaces)**.** A (closed, resp. open) *half-space* is specified by a basepoint $b \in \mathbb{Q}^2$ and a nonzero inward normal $n \in \mathbb{Q}^2$; a point $p$ belongs to the half-space iff $n \cdot (p - b) \geq 0$ (resp. $> 0$).

**Definition 6** (Half-space to the left of a directed edge)**.** Given distinct points $p_1, p_2 \in \mathbb{Q}^2$, the half-space strictly (resp. weakly) to the left of the directed segment $p_1 \to p_2$ uses basepoint $p_1$ and normal $R(p_2 - p_1)$.

**Definition 7** (Lines and intersection)**.** A *line* is a basepoint together with a nonzero direction vector. Two nonparallel lines have a unique intersection point, computed by Cramer's rule.

**Definition 8** (Moving a half-space inward)**.** Given $d, \tau > 0$, one may translate the basepoint of a half-space inward along its normal by an amount in $[d, d + \tau]$, obtaining a new half-space contained in the original. The scaling factor is realised rationally via Lemma 10.

**Definition 9** (Rational root with square between bounds)**.** Given $0 \leq \ell < u$ in $\mathbb{Q}$, there is an explicit rational $r > 0$ with $\ell \leq r^2 \leq u$.

**Lemma 10** (Correctness of `findRationalWithSquareBetween`)**.** *The rational* $r$ *returned by 9 satisfies* $r > 0$, $\ell \leq r^2$, *and* $r^2 \leq u$.

# Chapter 3

# Convex polygons

**Definition 11** (Non-degenerate polygon)**.** A *non-degenerate polygon* is a tuple consisting of a natural number $n \geq 3$, a function $V : \mathrm{Fin}\, n \to \mathbb{Q}^2$ giving the vertices in counterclockwise order, and a proof that $V$ is injective.

**Definition 12** (Convex polygon)**.** A *convex polygon* is a non-degenerate polygon in which, for every directed edge $V_i \to V_{i+1}$, every other vertex lies strictly to the left of the edge. Strictness excludes collinear triples and therefore makes every vertex an extreme point of the convex hull of $\{V_i\}$.

**Definition 13** (Edges as half-spaces)**.** Each edge of a convex polygon defines a closed half-space weakly to its left; the polygon coincides with the intersection of these half-spaces.

**Definition 14** (Containment and subset tests)**.** A point lies in a convex polygon iff it lies in every edge half-space. A convex polygon $P$ is a subset of $Q$ iff every vertex of $P$ lies in $Q$.

**Definition 15** (Convex hull of a list of rational points)**.** The function `convexHullRationalPoints` returns, from a finite list of rational points, a list of extreme points in counterclockwise order starting at the lexicographically smallest point. `ConvexPolygon.ofList` upgrades this to an `Option ConvexPolygon`, returning `none` when fewer than three extreme points remain.

**Lemma 16** (Convex hull vertices are distinct)**.** *The output of `convexHullRationalPoints` has no duplicates.*

**Definition 17** (Intersection of half-spaces as a polygon)**.** Given a list of closed half-spaces, the convex polygon defined by their intersection is obtained by taking all pairwise boundary intersections and retaining those satisfying every half-space constraint.

**Definition 18** (Shrinking a convex polygon)**.** Given $d, \tau > 0$, one shrinks a convex polygon by moving each edge inward by an amount in $[d, d + \tau]$ and intersecting the resulting half-spaces. The output is an `Option ConvexPolygon`, since the result may be empty.

**Definition 19** (Shoelace area)**.** The area of a convex polygon with vertex list $(V_0, \dots, V_{n-1})$ is

$$\mathrm{area}(P) = \frac{1}{2} \left| \sum_{i=0}^{n-1} (V_i)_0 (V_{i+1})_1 - (V_{i+1})_0 (V_i)_1 \right| .$$

# Chapter 4

# Planar isometries

**Definition 20** (Direct planar isometry)**.** A *direct isometry* is a tuple $(c, s, t)$ with $c, s \in \mathbb{Q}$, $t \in \mathbb{Q}^2$, and $c^2 + s^2 = 1$. It acts on $p \in \mathbb{Q}^2$ by

$$\text{apply}_{(c,s,t)}(p) = (c\,p_0 - s\,p_1,\ s\,p_0 + c\,p_1) + t.$$

**Lemma 21** (Direct isometries are bijections)**.** *The action of a direct isometry on $\mathbb{Q}^2$ is both injective and surjective.*

**Definition 22** (Applying an isometry to a polygon)**.** Applying a direct isometry vertex-wise to a convex polygon produces a convex polygon. Convexity is preserved because rotations preserve the cross products witnessing the "strictly left of" condition.

**Definition 23** (Composition of isometries)**.** Direct isometries are closed under composition, with $(c, s, t)_1 \circ (c, s, t)_2$ given by the usual product of rotation matrices and composed translation.

**Definition 24** (Rational unit circle grid)**.** For each pair of naturals $(a, b) \neq (0, 0)$, the formulas $c = (a^2-b^2)/(a^2+b^2)$ and $s = 2ab/(a^2+b^2)$ yield a rational point on the unit circle. Enumerating such pairs up to a resolution depending on a target *maximum angle change* yields a finite rational grid on the unit circle.

**Lemma 25** (Angle grid lies on the unit circle)**.** *Every $(c, s)$ produced by `angleGrid` satisfies $c^2 + s^2 = 1$.*

**Definition 26** (Isometry discretisation)**.** For a granularity $\epsilon > 0$, we form a finite list of direct isometries by pairing each angle from `angleGrid` with each translation on a rational grid intersected with the `LocationRange` (Definition 29). The angle resolution is scaled by the distance cutoff so that far-away points are still covered finely.

# Chapter 5

# Constants and the search region

**Definition 27** (Area threshold)**.** The area threshold is $A_* := 232240/10^6 = 0.232240$. A Moser set with area strictly less than $A_*$ would improve the current best lower bound.

**Definition 28** (Benchmark worms)**.** Three concrete convex polygons are fixed as benchmark worms:

- the isosceles right triangle with legs of length 1/2,
- the unit square with side 1/3,
- the right triangle with legs 1/3 and 2/3.

The *initial worm* is taken to be the isosceles right triangle; by convention every candidate Moser set in the working set contains an un-shifted copy of it. Its area is exactly 1/8.

**Definition 29** (Location range and distance cutoff)**.** Let $\sigma := 4A_*$. The *location range* is the square $[-\sigma, \sigma]^2$, which upper-bounds the positions where a point of a working-set polygon may lie without already exceeding the area threshold (via a triangle formed with the initial worm). The *distance cutoff* is $\sigma \cdot \overline{\sqrt{2}}$, where $\overline{\sqrt{2}} = 1414213562373095/10^{15}$ is a rational upper bound on $\sqrt{2}$.

**Proposition 30** (Points outside the location range are excluded)**.** *Any convex polygon containing the initial worm as well as a point outside the location range has area strictly greater than $A_*$.*

# Chapter 6

# Worms

**Definition 31** (Newton-style rational square root)**.** Given $s \geq 0$ and $\epsilon > 0$, the Newton iteration $x_{n+1} = (x_n + s/x_n)/2$ produces a rational $r$ with $|r - \sqrt{s}| < \epsilon$. Applied componentwise, this yields a rational approximation `distanceApprox` of the Euclidean distance between two rational points.

**Definition 32** (Worm)**.** A *worm* is a polygonal path with at least two rational vertices. Its approximate total length is the sum of distance approximations along consecutive vertices, with per-segment error budget divided equally so the total error remains below a given $\epsilon$.

**Definition 33** (Scaling a worm, unit worm)**.** A worm can be scaled by any rational factor; scaling by $1/\ell$ where $\ell$ is the approximate length brings the worm to approximately unit length. A *unit worm* bundles a worm together with a uniform bound ensuring the approximate length tends to 1 as $\epsilon \to 0$.

**Definition 34** (Convex hull of a worm)**.** The convex hull of a worm is a convex polygon (when the worm has at least three non-collinear vertices); it summarises all rigid motions of the worm that embed it into a convex target.

# Chapter 7

# Moser sets

**Definition 35** (Moser set)**.** A set $M \subseteq \mathbb{R}^2$ is a *Moser set* if for every worm $w$ of unit length there exists a direct isometry $\varphi$ with $\varphi(w) \subseteq M$.

**Definition 36** (Approximate / computational Moser set)**.** Given a finite list of candidate worms $W$ and a finite list of isometries $I$, a set $M$ is a $(W, I)$-*approximate Moser set* if for every $w \in W$ there exists $\varphi \in I$ with $\varphi(w) \subseteq M$.

# Chapter 8

# Working set algorithm

The computational approach maintains a list of convex polygons, the *working set*, with the invariant that every Moser set of area below the threshold contains (a rigid motion of) at least one polygon in the list.

**Definition 37** (Working set)**.** A *working set* is a list of convex polygons. Intended invariants:

1. every polygon in the list contains a copy of the initial worm (Definition 28) under some direct isometry;
2. every Moser set of area strictly less than $A_*$ contains a copy of some polygon in the list under some direct isometry.

Invariant 1 is currently a convention maintained by construction; Invariant 2 is the core correctness statement the algorithm preserves.

**Definition 38** (Initial working set)**.** The initial working set contains exactly the initial worm.

**Definition 39** (Minimum area)**.** The minimum area in a working set is the minimum of the shoelace areas of its polygons (taken as 0 on the empty set).

## 8.1 Pruning operations

**Definition 40** (Big set removal)**.** Remove every polygon whose area strictly exceeds $A_*$. This preserves Invariant 2 because any Moser set of area below $A_*$ cannot have a polygon of strictly larger area as a subset.

**Proposition 41** (Big set removal preserves the working set invariant)**.** *If a working set satisfies Invariant 2 of Definition 37, so does the result of big set removal.*

**Definition 42** (Superset removal)**.** Remove every polygon $p$ in the list for which some distinct polygon $q$ in the list is a subset of $p$. The smaller $q$ already witnesses Invariant 2 whenever $p$ does.

**Proposition 43** (Superset removal preserves the working set invariant)**.** *If a working set satisfies Invariant 2, so does the result of superset removal.*

**Definition 44** (Worm adding)**.** Given an additional worm presented as a convex hull, for each polygon $p$ in the current working set and for each isometry $\varphi$ in the discretisation, form the convex hull of $p$ together with the image under $\varphi$ of the *shrunk* hull (to absorb discretisation error). The new working set collects all such unions. This step encodes the requirement that a Moser set must contain *this* worm in some pose.

**Proposition 45** (Worm adding preserves the working set invariant)**.** *If the isometry discretisation is fine enough relative to the shrink amount, then adding a worm preserves Invariant 2 of Definition 37.*

**Definition 46** (Cleanup and combined add)**.** `cleanup` is the composition of big-set removal and superset removal. `addWormAndCleanup` adds a worm and then cleans up.

# Chapter 9

# Lower bound theorem

The end goal of the working set iteration is the following claim.

**Theorem 47** (Computational lower bound)**.** *There exists a finite list of worms $W$ and a sequence of working-set updates that, starting from Definition 38 and adding the worms in $W$ in turn using Definition 46, terminates with the empty working set. Consequently the area of every Moser set is at least $A_*$:*

$$\inf\{\mathrm{area}(M) : M \text{ is a Moser set}\} \geq A_*.$$

*Proof.* Any Moser set $M$ of area $< A_*$ must contain, in some pose, every polygon ever present in a working set produced by `addWormAndCleanup`, by Propositions 41, 43, and 45. If the final working set is empty, this is a contradiction. ☐

**Proposition 48** (Terminating sequence of worms)**.** *A finite list of worms ending the iteration with the empty working set can be found by computer search, e.g. starting from the benchmark worms of Definition 28 and augmenting with further worms arising from the enumeration scheme in `Moser.Worm.Enumeration` (currently stubbed).*